53. (a) Let \( f \) be a continuous function of one variable. Show that if \( f \) has two local maxima, then \( f \) must also have a local minimum.

(b) The analogue of part (a) does not necessarily hold for continuous functions of more than one variable, as we now see. Consider the function

\[
f(x, y) = 2 - (x y^2 - y - 1)^2 - (y^2 - 1)^2.
\]

Show that \( f \) has just two critical points—and that both of them are local maxima.

(c) Use a computer to graph the function \( f \) in part (b).

4.3 Lagrange Multipliers

Constrained Extrema

Frequently, when working with applications of calculus, you will find that you do not need simply to maximize or minimize a function but that you must do so subject to one or more additional constraints that depend on the specifics of the situation. The following example is a typical situation:

**EXAMPLE 1** An open rectangular box is to be manufactured having a (fixed) volume of 4 ft\(^3\). What dimensions should the box have so as to minimize the amount of material used to make it?

We’ll let the three dimensions of the box be independent variables \( x, y, \) and \( z \), shown in Figure 4.25. To determine how to use as little material as possible, we need to minimize the surface area function \( A \) given by

\[
A(x, y, z) = 2xy + 2yz + xz,
\]

where \( x > 0, y > 0, \) and \( z > 0 \). This function has neither minimum nor maximum. However, we have not yet made use of the fact that the volume is to be maintained at a constant 4 ft\(^3\). This fact provides a constraint equation,

\[
V(x, y, z) = xyz = 4.
\]

The constraint is absolutely essential if we are to solve the problem. In particular, the constraint enables us to solve for \( z \) in terms of \( x \) and \( y \):

\[
z = \frac{4}{xy}.
\]

We can thus create a new area function of only two variables:

\[
a(x, y) = A \left( x, y, \frac{4}{xy} \right)
\]

\[
= 2xy + 2y \left( \frac{4}{xy} \right) + x \left( \frac{4}{xy} \right)
\]

\[
= 2xy + \frac{8}{x} + \frac{4}{y}.
\]

Now we can find the critical points of \( a \) by setting \( Da \) equal to 0:

\[
\begin{align*}
\frac{\partial a}{\partial x} &= 2y - \frac{8}{x^2} = 0 \\
\frac{\partial a}{\partial y} &= 2x - \frac{4}{y^2} = 0.
\end{align*}
\]